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A NECESSARY AND SUFFICIENT CONDITION FOR A CLASS ($\mathcal{L}$) TO BE A CLASS ($\mathfrak{D}$)

~~by~~

Paul Alexandroff and Paul Urysohn

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THEORY OF SETS. -- A necessary and sufficient condition for a class ($\mathcal{L}$) to be a class ($\mathcal{D}$) [1].

Note by Paul Alexandroff and Paul Urysohn, presented by Henri Lebesgue.

M. Frechet[2] was the first to formulate explicitly the problem of indicating the conditions required for a class ($\mathcal{L}$) to be a class ($\mathcal{D}$), i.e., to be able to determine in a class ($\mathcal{L}$) a distance such that the limiting relations specified in advance would be the same as those to which it gave rise. This problem, the literature on which has already been enriched by the important contributions of various authors (Hedrick, Frechet, Chittenden, Moore, Victoris, Urysohn, Alexandroff) who solved it in particular cases, is equivalent to the following problem: under what conditions would a topological space[3] be a metric space? In effect, each metric space would be /1274

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[1] A class ($\mathcal{L}$) is a class where the limit is defined (M. Frechet, Palermo Rend., 1906). It will be termed ($\mathcal{S}$) if every derived set is closed. A class ($\mathcal{E}$) is a class in which a distance function is defined, i.e., a function

$$\rho(x,y) = \rho(y,x) \geqq 0$$

which vanishes only when $x = y$. This class will be termed a ($\mathcal{E}_r$) class if the distance function is regular, i.e., if there exists a function $f(t)$ such that the relations $\rho(x,y) < t$ and $\rho(y,z) < t$ imply $\rho(x,z) < f(t)$. The distance defining a class ($\mathcal{D}$) is a regular distance function verifying the more restrictive inequality $\rho(x,z) \leqq \rho(x,y) + \rho(y,z)$. F. Hausdorff calls the ($\mathcal{D}$) classes metric spaces. We owe to E. Chittenden (Trans. Am. Math. Soc. 18, 161 (1917) the important result that every ($\mathcal{E}_r$) class is a ($\mathcal{D}$) class.

[2] Trans. Am. Math. Soc. 19, 55 (1918); moreover, Frechet has been dealing with this problem as early as 1913 (Trans. Am. Math. Soc., 14, 320 (1913)).

[3] Hausdorff, Grundzüge der Mengenlehre (Theory of Sets), Leipzig, Chapter VII, Section 1, 1914.

[redacted]

regarded as a topological space and as a class ($\mathcal{L}$) [just as a class ($\mathcal{S}$)] and the conditions required for a topological space to be a class ($\mathcal{L}$) and vice versa[4] could be indicated with ease.

Definitions. I. Let $\{V_n\}$ be a sequence of domains each of which contains the point ξ; we shall say that the sequence determines this point in the given topological space E if each domain G containing ξ contains at least one domain V_n. /1275

II. We shall say that a system π of domains covers the space E if each point ξ of E belongs to at least one of the domains of the system π.

III. Let π_1 and π_2 be two systems of domains each of which covers E; we shall say that π_2 is inscribed in π_1 if to each pair V_2 and W_2 of domains belonging to π_2 and having points in common there corresponds a domain V_1 of π_1 which contains both of them.

IV. Let $\{\pi_1, \pi_2, \ldots, \pi_n, \ldots\}$ be a sequence of systems covering the space; we shall say that this sequence is a complete chain if the following condition is met: let ξ be any point of E and let $V_1, V_2, \ldots, V_n, \ldots$ be domains containing ξ and belonging to $\pi_1, \pi_2, \ldots, \pi_n, \ldots$, respectively; in this case the sequence $\{V_n\}$ determines point ξ in E.

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[4] For example, for a ($\mathcal{L}$) class to be a topological space, it is necessary and sufficient that the following three conditions be met:

1. It is a ($\mathcal{S}$) class.
2. There exist, for every pair of elements, two domains [Open sets, in modern terminology -- Transl. note] (i.e., two sets complementary to closed sets which are disjoint and which contain the two specified elements, respectively.
3. If every partial sequence σ_1 of a sequence σ contains a subsequence σ_2 which converges on the element a, then the total sequence σ converges on the same element a.

V. A complete chain $\{\pi_1, \pi_2, \ldots, \pi_n, \ldots\}$ will be termed <u>regular</u> if, for every n, π_{n+1} is inscribed in π_n.

<u>Theorem</u>. For a topological space to be considered a metric space, it is necessary and sufficient that there exist a complete regular chain.

<u>Proof</u>. To see the necessity of this condition, it suffices to call π_n the system formed of all the <u>spheres</u> of radius 2^{-n} [a sphere centered at x and of radius ϵ being, by definition, the set of points y such that $\rho(x,y) < \epsilon$].

Let us now show that a topological space E admitting a complete chain $\{\pi_1, \pi_2, \ldots, \pi_n, \ldots\}$ is a $(\mathcal{E})$ class. Since x and y are two arbitrary points of E, we define their distance function $\rho(x,y)$ as follows: if there does not exist any domain of π which contains both of them, we put $\rho(x,y) = 1$; otherwise, let n be the first integer such that no domain of π_{n+1} contains these two points simultaneously: we now put $\rho(x,y) = 2^{-n}$. The point is to prove that this distance function is in accord with the limiting relations, i.e.:

1. Corresponding to each point ξ of E and to every number $\epsilon > 0$ we have a domain G containing ξ and all the points x of which satisfy the inequality $\rho(\xi,x) < \epsilon$. In effect, let n be the first integer such that $2^{-n} < \epsilon$, and let $V_1, V_2, \ldots, V_n$ be domains belonging to $\pi_1, \pi_2, \ldots, \pi_n$, respectively, and each containing the point ξ; it then suffices to denote their union as G;

2. Corresponding to each point ξ of E and to every domain G containing ξ, we have an $\epsilon > 0$ such that every point x satisfying $\rho(\xi,x) < \epsilon$ is situated in G. Indeed, contrariwise there would exist points x outside of G such that $\rho(\xi,x)$ would be arbitrarily small; there would then be in each π_n at least one domain V_n containing ξ without being contained in G: and this is in contradiction with the definition of the complete chains.

3. Now if the complete chain is a regular complete chain, /1276 the two inequalities $\rho(x,y) \leqq 2^{-n}$ and $\rho(y,z) \leqq 2^{-n}$ obviously imply $\rho(x,z) \leqq 2^{-(n-1)}$, i.e., the distance function is regular; according to Chittenden's theorem, then, E is a metric space. Q.E.D.

Note added in proof. M. Frechet has been so kind as to inform us that the condition requiring that a class $(\mathcal{L})$ be a $(\mathcal{O})$ class could be stated in a much simpler way than that involving the use of topological spaces. In effect, our theorem pertinent to these spaces (as well as the above proof) also applies directly to the $(\mathcal{S})$ classes, verifying condition 3 (see note 4), and even applies, more generally, to classes $(\mathcal{H})$.

FARADAY TRANSLATIONS
15 PARK ROW
NEW YORK 38, N.Y.